Hankel and Loewner Matrices

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ABSTRACT

Mutual relations between the Hankel, Toeplitz, Bézout, and Loewner matrices as well as further connections to rational interpolation and projective geometry are investigated.

INTRODUCTION

We intend to give a survey of known results as well as to add a few new results on mutual relations between the classes of Hankel, Toeplitz, Bézout, and Loewner matrices, as well as mentioning connections with other fields such as rational interpolation, reciprocal differences, and the generalization of the Poncelet theorem of projective geometry.

1. HANKEL, TOEPLITZ, AND BÉZOUT MATRICES

As is well known, Hankel matrices [5] are square matrices of the form (α_{i+k}), $i,k=0,\dots,n-1$, where $\alpha_0,\alpha_1,\dots,\alpha_{2n-2}$ are, as we shall always assume, complex numbers. Toeplitz matrices of order n have the form (τ_{i-k}), $i,k=0,\dots,n-1$, where $\tau_{-(n-1)},\dots,\tau_{-1},\tau_0,\tau_1,\dots,\tau_{n-1}$ are again complex. There are obvious relations between the two classes:

THEOREM 1. *Let J be the $n\times n$ matrix*

$$J=\begin{pmatrix}0&0&\cdots&0&1\\0&0&\cdots&1&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\1&0&\cdots&0&0\end{pmatrix}.$$

Then A is Hankel iff AJ is Toeplitz (or iff JA is Toeplitz).

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The Bézout matrices, also called Bézoutians [9], are characterized as square matrices, say B, for which there exists a pair of (complex) polynomials $f(x)$, $g(x)$ with $\max(\deg f, \deg g) = n$, the order of B, such that for

$$X = (1, x, \ldots, x^{n-1})^T, \qquad Y = (1, y, \ldots, y^{n-1})^T, \tag{1}$$

we have

$$X^T B Y = \frac{f(x)g(y) - f(y)g(x)}{x - y} \tag{2}$$

(the right-hand side is indeed a polynomial in x and y).

As is well known, $\det B$ is the Bézout form of the resultant of the polynomials f and g. Consequently:

THEOREM 2. *The Bézoutian B is nonsingular iff the polynomials f and g are relatively prime. Moreover, if we denote the corresponding B in* (2) *as $B(f, g)$, we have*

$$B(\alpha f + \beta g, \gamma f + \delta g) = (\alpha\delta - \beta\gamma) B(f, g). \tag{3}$$

In the sequel, Vandermonde matrices will be of basic importance. For complex $t_1, \ldots, t_n$, the Vandermonde matrix $V(t)$ [or, more precisely, $V(t_1, \ldots, t_n)$] is the square matrix

$$\begin{aligned} V(t) &= (t_i^k), \qquad i = 1, \ldots, n, \quad k = 0, \ldots, n-1 \\ &= \begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & t_n & \cdots & t_n^{n-1} \end{pmatrix}. \end{aligned} \tag{4}$$

As is well known,

$$\det V(t) = \prod_{1 \leqslant i < k \leqslant n} (t_i - t_k). \tag{5}$$

THEOREM 3. *The matrix $V(t)$ is nonsingular iff $t_1, \ldots, t_n$ are mutually distinct. Moreover, if*

$$f_i(x) = \prod_{j \neq i} (x - t_j) \tag{6}$$

and the $n \times n$ *matrix* $W(t)$ $(=W(t_1,\dots,t_n))$ *is defined by* (X *as in* (1))

$$W(t)X = (f_1(x),\dots,f_n(x))^T, \tag{7}$$

then

$$W(t)V^T(t) = D(t) = \operatorname{diag}(f_1(t_1),\dots,f_n(t_n)). \tag{8}$$

Consequently, if $t_1,\dots,t_n$ *are distinct,*

$$[V(t)]^{-1} = W^T(t)D^{-1}(t). \tag{9}$$

In the next theorem, which comprises some known and some new facts about the Hankel and Bézout matrices, we shall need some notation.

If $H=(h_{i+k})$, $i,k=0,\dots,n-1$, is Hankel, then we denote by $\hat{H}$ the $(n-1)\times(n+1)$ matrix

$$\hat{H} = (h_{i+k}), \qquad i=0,\dots,n-2, \quad k=0,\dots,n, \tag{10}$$

and by $[H]$ the row vector

$$[H] = (h_0, h_1,\dots,h_{2n-2}). \tag{11}$$

If $f(x) \equiv f_0 + f_1x + \cdots + f_nx^n$, then

$$[f]_n = (f_0, f_1,\dots,f_n)^T, \tag{12}$$

$$C_n(f) = \begin{pmatrix} 0 & f_n & 0 & \cdots & 0 \\ 0 & 0 & f_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & f_n \\ -f_0 & -f_1 & -f_2 & \cdots & -f_{n-1} \end{pmatrix}. \tag{13}$$

In (12) and (13), f_n may also be zero.

THEOREM 4. *Let* A *be a nonsingular* $n\times n$ *complex matrix; let* f, g *be relatively prime polynomials in one variable such that* $\max(\deg f, \deg g) = n$. *Then the following are equivalent:*

(i) A *is Hankel and (using the notation* (10), (12))

$$\hat{A}[f]_n = 0, \qquad \hat{A}[g]_n = 0;$$

(ii) *A is Hankel and (using the notation* (11)*)*

$$[A]\begin{pmatrix} f_0 & 0 & \cdots & 0 & g_0 & 0 & \cdots & 0 \\ f_1 & f_0 & \cdots & 0 & g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{n-1} & \cdots & f_2 & g_n & g_{n-1} & \cdots & g_2 \\ 0 & f_n & \cdots & f_3 & 0 & g_n & \cdots & g_3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f_n & 0 & 0 & \cdots & g_n \end{pmatrix} = 0;$$

(iii) *for* $C_n(f)$ *from* (13)

$$AC_n^T(f) = C_n(f)A, \qquad AC_n^T(g) = C_n(g)A;$$

(iv) *whenever* α, β *are numbers for which* $\alpha f(x) + \beta g(x)$ *has* n *simple roots* $t_1, \dots, t_n$, *then*

$$A = V^T(t)CV(t) \tag{14}$$

for some diagonal matrix C; *equivalently, if* $C = \mathrm{diag}(c_i)$, *then* $A = (\alpha_{i+k})$ *and*

$$\alpha_s = \sum_i c_i t_i^s, \qquad s = 0, \dots, 2n-2; \tag{15}$$

(v) *A is Hankel*: $A = (\alpha_{i+k})$, $i, k = 0, \dots, n-1$, *and there exist numbers* $\rho_1 \neq 0$, $\rho_2 \neq 0$, ξ_1, ξ_2 *(maybe infinite)*, $\xi_1 \neq \xi_2$, *such that*

$$f(x) = \rho_1 A(x, \xi_1), \qquad g(x) = \rho_2 A(x, \xi_2),$$

where

$$A(x,\xi) = \det\begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n & x \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n & \cdots & \alpha_{2n-3} & \alpha_{2n-2} & x^{n-1} \\ \alpha_n & \alpha_{n+1} & \cdots & \alpha_{2n-2} & \xi & x^n \end{pmatrix} \tag{16}$$

if ξ is finite and

$$A(x,\infty)=\det\begin{pmatrix}\alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & 1\\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & x\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_{n-1} & \alpha_n & \cdots & \alpha_{2n-3} & x^{n-1}\end{pmatrix}; \qquad (16')$$

(vi) *whenever $\alpha,\beta,\gamma,\delta$ are numbers for which $h(x)\equiv\alpha f(x)+\beta g(x)$ has n simple roots, say $t_1,\dots,t_n$, and $\alpha\delta-\beta\gamma\neq 0$, then*

$$A=\sigma V^T(t)M^{-1}V(t) \qquad (17)$$

for some $\sigma\neq 0$, where $M=\operatorname{diag}(m_i)$ with

$$m_i=[\gamma f(t_i)+\delta g(t_i)]\prod_{j\neq i}(t_i-t_j), \qquad i=1,\dots,n; \qquad (18)$$

(vii) *there exist numbers $\alpha,\beta,\gamma,\delta$ such that $h(x)\equiv\alpha f(x)+\beta g(x)$ has n simple roots $t_1,\dots,t_n$, $\alpha\delta-\beta\gamma\neq 0$, and*

$$A=V^T(t)M^{-1}V(t),$$

where $M=\operatorname{diag}(m_i)$, m_i fulfilling (18);

(viii) *there exists a $\rho\neq 0$ such that ($B(f,g)$ defined in* (2))

$$A^{-1}=\rho B(f,g).$$

Proof. We shall prove that (i) → (ii) → (iii) → (iv) → (v) → (i), (iv) → (vi) → (vii) → (viii) → (iv).

(i) → (ii): Immediate.

(ii) → (iii): Since $A=(\alpha_{i+k})$ is Hankel,

$$AC_n^T(f)$$
$$=\begin{pmatrix}\alpha_1 f_n & \alpha_2 f_n & \cdots & \alpha_{n-1}f_n & -(\alpha_0 f_0+\cdots+\alpha_{n-1}f_{n-1})\\ \alpha_2 f_n & \alpha_3 f_n & \cdots & \alpha_n f_n & -(\alpha_1 f_0+\cdots+\alpha_n f_{n-1})\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_n f_n & \alpha_{n+1}f_n & \cdots & \alpha_{2n-2}f_n & -(\alpha_{n-1}t_0+\cdots+\alpha_{2n-2}f_{n-1})\end{pmatrix}.$$

By (ii), $AC_n^T(f)$ is symmetric, so that

$$\begin{aligned} AC_n^T(f) &= C_n(f)A^T \\ &= C_n(f)A. \end{aligned}$$

Similarly for g.

(iii) → (iv): Suppose (iii). For some α, β, let $h(x) \equiv \alpha f(x) + \beta g(x)$ have n simple roots $t_1, \ldots, t_n$. Then, $h_n \neq 0$ being the coefficient at x^n in $h(x)$,

$$C_n(h)V^T(t) = h_n V^T(t)D \tag{19}$$

where $D = \operatorname{diag}(t_i)$. Therefore, using (19) twice,

$$\begin{aligned} D[V^T(t)]^{-1}A[V(t)]^{-1} &= h_n^{-1}[V^T(t)]^{-1}C_n(h)A[V(T)]^{-1} \\ &= h_n^{-1}[V^T(t)]^{-1}AC_n^T(h)[V(t)]^{-1} \\ &= [V^T(t)]^{-1}A[V(t)]^{-1}D. \end{aligned}$$

Since D is diagonal with distinct diagonal entries, $C = [V^T(t))^{-1}A(V(t)]^{-1}$ is again diagonal and (14) holds. (15) is equivalent with (14).

(iv) → (v): To prove this, let us state a simple lemma the proof of which is left to the reader:

LEMMA 5. *If $f(x)$ and $g(x)$ are relatively prime complex polynomials with* $\max(\deg f, \deg g) = n \geqslant 1$, *then there exists a number (and even an infinite number of such) λ for which $f(x) + \lambda g(x)$ has n simple roots.*

Let $h_i = f + \lambda_i g$, $i = 1,2$, $\lambda_1 \neq \lambda_2$, be polynomials each of which having n simple roots, let $t_1, \ldots, t_n$ be the roots of h_1. By (iv), A is Hankel, $A = (\alpha_{i+k})$ where (15) holds. Set

$$\eta_1 = \sum_{i=1}^{n} c_i t_i^{2n-1}.$$

It is easy to check that $A(x, \eta_1) = 0$ for $x = t_1, \ldots, t_n$. Consequently

$$f + \lambda_1 g \equiv \tau_1 A(x, \eta_1) \qquad \text{for some} \quad \tau_1 \neq 0.$$

Similarly,

$$f + \lambda_2 g \equiv \tau_2 A(x, \eta_2), \qquad \tau_2 \neq 0.$$

Thus g has the asserted form with ξ_2 finite if $\tau_1 \neq \tau_2$ and infinite if $\tau_1 = \tau_2$. Similarly for f.

(v) → (i): For $k = 0, \ldots, n-2$, the kth component of $\hat{A}[f]_n$ is obtained by substituting $\alpha_{k+n}, \alpha_{k+n-1}, \ldots, \alpha_{k+1}, \alpha_k$ for $x^n, x^{n-1}, \ldots, x, 1$ into $f(x)$. Therefore, (v) implies, if ξ_1 is finite,

$$(\hat{A}[f]_n)_k = \rho_1 \det \begin{pmatrix} \alpha_0 & \cdots & \alpha_{n-2} & \alpha_{n-1} & \alpha_k \\ \alpha_1 & \cdots & \alpha_{n-1} & \alpha_n & \alpha_{k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \cdots & \alpha_{2n-2} & \xi_1 & \alpha_{k+n} \end{pmatrix},$$

which is zero. The same is true if ξ_1 is infinite. Similarly, $\hat{A}[g]_n = 0$.

(iv) → (vi): We know that under the assumptions of (iv), (14) holds, where $t_1, \ldots, t_n$ are the (simple) roots of $h(x)$ and $C = \operatorname{diag}(c_i)$. To prove that $\sigma \neq 0$ exists for which

$$c_i = \sigma m_i^{-1}, \qquad i = 1, \ldots, n, \tag{20}$$

where m_i is defined in (18), we shall use the fact that by the proved equivalence of (iv) with (ii),

$$[A] \begin{pmatrix} k_0 & 0 & \cdots & 0 \\ k_1 & k_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ k_n & k_{n-1} & \cdots & \\ 0 & k_n & \cdots & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k_n \end{pmatrix} = 0, \tag{21}$$

where $k(x)$ $(\equiv k_0 + k_1 x + \cdots + k_n x^n) \equiv \gamma f(x) + \delta g(x)$ and $[A]$ is defined in (11).

By the equivalence of (14) and (15),

$$[A] = (c_1, \ldots, c_n) \begin{pmatrix} 1 & t_1 & \cdots & t_1^{2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & t_n & \cdots & t_n^{2n-2} \end{pmatrix},$$

so that (21) implies

$$(c_1k(t_1), c_2k(t_2), \ldots, c_nk(t_n))\begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & t_n & \cdots & t_n^{n-2} \end{pmatrix} = 0.$$

Since every solution of

$$(x_1, \ldots, x_n)\begin{pmatrix} 1 & t_1 & \cdots & t_1^{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & t_n & \cdots & t_n^{n-2} \end{pmatrix} = 0$$

has the form

$$x_i = \frac{\sigma}{\prod_{j \neq i}(t_i - t_j)},$$

we obtain (20). Here, $\sigma \neq 0$, since A is nonsingular.

(vi) $\to$ (vii): Follows easily using Lemma 5.

(vii) $\to$ (viii): By Theorem 2, it suffices to prove this implication under the assumption that $\alpha = 1$, $\delta \neq 0$, $\beta = \gamma = 0$. Thus let $t_1, \ldots, t_n$ be simple roots of f; let A satisfy (17) for $\sigma = 1$ and (18).

Consequently, X and Y being defined as in (1), we have by (9)

$$\begin{aligned} X^TA^{-1}Y &= X^T[V(t)]^{-1}M[V^T(t)]^{-1}Y \\ &= X^TW^T(t)D^{-1}MD^{-1}W(t)Y \end{aligned} \tag{22}$$

for $D = \operatorname{diag}(f_i(t_i))$, $f_i(x) = \prod_{j \neq i}(x - t_j)$. Thus

$$X^TA^{-1}Y = \left(\frac{f_1(x)}{f_1(t_1)}, \ldots, \frac{f_n(x)}{f_n(t_n)}\right)M\left(\frac{f_y(y)}{f_1(t_1)}, \ldots, \frac{f_n(y)}{f_n(t_n)}\right)^T.$$

Since $m_i = g(t_i)/f_i(t_i)$, we have

$$X^TA^{-1}Y = \sum_{i=1}^{n} f_i(x)f_i(y)\frac{g(t_i)}{f_i(t_i)}. \tag{23}$$

On the other hand, if we express

$$g(x)=\gamma_0 f(x)+\sum_{i=1}^{n}\gamma_i f_i(x),$$

which is always possible, we obtain by setting $x=t_i$

$$g(t_i)=\gamma_i f_i(t_i).$$

Then

$$X^T B(f,g)Y=\sum_{i=1}^{n}\gamma_i X^T B(f,f_i)Y.$$

However, $f(x)=\tau(x-t_i)f_i(x)$, $\tau\neq 0$, so that

$$\begin{aligned} X^T B(f,f_i)Y &= \frac{f(x)f_i(y)-f(y)f_i(x)}{x-y} \\ &= \tau f_i(x)f_i(y). \end{aligned}$$

Thus

$$X^T B(f,g)Y=\tau\sum f_i(x)f_i(y)\frac{g(t_i)}{f_i(t_i)}. \tag{24}$$

Comparing this with (23) yields (viii).

(viii)$\to$(iv): Let $A^{-1}=\rho B(f,g)$, $\rho\neq 0$. Let $h(x)\equiv\alpha f(x)+\beta g(x)$ have simple roots $t_1,\dots,t_n$. W.l.o.g. we can assume $\beta=0$.

By (24),

$$X^T A^{-1}Y=\rho\tau\sum_{i=1}^{n} f_i(x)f_i(y)\frac{g(t_i)}{f_i(t_i)}.$$

Going back from (23) to (22), we obtain

$$X^T A^{-1}Y=X^T[V(t)]^{-1}M[V^T(t)]^{-1}$$

with M diagonal nonsingular.

Consequently, $A = V^T(t)M^{-1}V(t)$ with M^{-1} diagonal. The proof is complete. ■

REMARK. One can easily show that for a nonsingular $n \times n$ Hankel matrix A the linear space of polynomials f of degree at most n satisfying $\hat{A}[f]_n = 0$ has dimension 2. Moreover, any two linearly independent polynomials in this space are relatively prime. We shall call a pencil $\alpha f + \beta g$ of polynomials a proper n-pencil if the maximum degree is n and if some two (and then any two linearly independent) polynomials are relatively prime.

By Theorem 2 and (viii) of Theorem 4, the following corollaries hold:

COROLLARY 6. *There is a one-to-one correspondence between the class of nonzero multiples of a nonsingular $n \times n$ Hankel matrix and proper n-pencils of polynomials.*

COROLLARY 7 (Lander [10]). *The inverse of a nonsingular Hankel matrix is a Bézout matrix and conversely.*

COROLLARY 8. *A nonsingular matrix B is a Bézout matrix corresponding to linearly independent polynomials f, g (i.e. $B = \lambda B(f, g)$) iff both*

$$BC_n(f) = C_n^T(f)B, \qquad BC_n(g) = C_n^T(g)B$$

are fulfilled.

Corollary 7 has a practical application (Trench [13], Gohberg and Krupnik [6]), that for obtaining the inverse of a Hankel matrix A it suffices to solve just two linear systems with the matrix A. We present here a formula which is always applicable (it does not contain division) and whose Toeplitz equivalent is due to Heinig and Rost [8].

THEOREM 9. *Let $A = (\alpha_{i+k})$, $i, k = 0, \dots, n-1$, be a nonsingular Hankel matrix. Let $u = (u_0, \dots, u_{n-1})^T$, $v = (v_0, \dots, v_{n-1})^T$ be solutions of*

$$Au = p, \qquad Av = q,$$

where

$$p = (\alpha_n, \dots, \alpha_{2n-2}, 0)^T, \qquad q = (0, 0, \dots, 0, 1)^T.$$

Then

$$A^{-1} = B(f, g)$$

for

$$f(x) \equiv x^n - \sum_{i=0}^{n-1} u_i x^i, \qquad g(x) \equiv \sum_{i=0}^{n-1} v_i x^i.$$

In other words,

$$A^{-1} = \begin{pmatrix} v_1 & v_2 & \cdots & v_{n-1} & 0 \\ v_2 & v_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 & u_1 & \cdots & u_{n-1} \\ 0 & u_0 & \cdots & u_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_0 \end{pmatrix}$$

$$- \begin{pmatrix} u_1 & u_2 & \cdots & u_{n-1} & -1 \\ u_2 & u_3 & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{n-1} & -1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} v_0 & v_1 & \cdots & v_{n-1} \\ 0 & v_0 & \cdots & v_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & v_0 \end{pmatrix}.$$

Hankel matrices have also a close connection with a generalization of the classical Poncelet theorem of projective geometry [3]:

THEOREM 10. *Let C be the rational normal curve in the projective $(n-1)$-space P_{n-1} ($x_0, \dots, x_{n-1}$ being projective coordinates in P_{n-1})*

$$x_0 = t_0^{n-1}, \qquad x_1 = t_0^{n-2} t_1, \dots, \qquad x_{n-1} = t_1^{n-1},$$

or, in the nonhomogeneous form, considering infinity as a choice of the parameter t,

$$x_0 = 1, \qquad x_1 = t, \dots, \qquad x_{n-1} = t^{n-1}.$$

Then the matrix $A = (a_{ik})$ of a dual nonsingular quadric $Q \equiv \sum_{i,k=0}^{n-1} a_{ik} \xi_i \xi_k = 0$ is Hankel iff there exists an $(n-1)$-simplex all of whose n vertices are points of C and which is polar with respect to Q (i.e., the polar hyperplane of each vertex with respect to Q passes through the remaining $n-1$ vertices).

Proof. Let $A^{-1} = (\alpha_{ik})$ be the inverse of A; let $t_1, \dots, t_n$ be the nonhomogeneous parameters of vertices of such a simplex Σ as points of C. Then the polarity of Σ yields

$$\sum_{i,k=0}^{n-1} \alpha_{ik} t_r^k t_s^i = 0 \qquad \text{for all} \quad r, s, \quad r \neq s. \tag{25}$$

This means that the matrix

$$L = V(t) A^{-1} V^T(t)$$

is diagonal, so that

$$A = V^T(t) L^{-1} V(t) \tag{26}$$

is Hankel.

Conversely, if A is Hankel, then there exists an n-tuple $t_1, \dots, t_n$ and a diagonal nonsingular matrix L such that (26) holds. But then (25) implies the corresponding $(n-1)$-simplex is polar with respect to Q. ■

Theorem 4 has then the following generalization of the Poncelet theorem as its corollary:

COROLLARY 11. *If there is one $(n-1)$-simplex inscribed in a rational normal curve in a projective $(n-1)$-space and polar with respect to a nonsingular quadric Q, then there exist infinitely many such simplices.*

2. LOEWNER MATRICES

Let $y_1, \dots, y_n, z_1, \dots, z_n$ be fixed distinct complex numbers. We shall denote by $\mathscr{L}(y, z)$ the set of all $n \times n$ matrices of the form

$$\left(\frac{c_i - d_j}{y_i - z_j} \right), \qquad i, j = 1, \dots, n,$$

where $c_1, \dots, c_n, d_1, \dots, d_n$ are complex numbers, and call such matrices Loewner matrices [2, 11]. Belevitch [1] speaks of DD-matrices.

Matrices in $\mathscr{L}(y, z)$ form clearly a linear subspace in the space of all complex $n \times n$ matrices. Its dimension is $2n - 1$, since addition of a constant to all $2n$ parameters c_i, d_j leads to the same Loewner matrix. The set of all $n \times n$ Hankel matrices also forms a linear subspace with dimension $2n-1$. The relation between the Hankel and Loewner matrices is, however, even closer. In the following theorem, which we shall present without proof [1, 4], we shall use the notation of (4)–(8). Moreover, we shall denote by $V(y,z)$ the $2n \times 2n$ Vandermonde matrix $V(y_1,\dots,y_n,z_1,\dots,z_n)$, and similarly for $W(y,z)$.

THEOREM 12. *Let* $A=(\alpha_{i+k})$, $i,k=0,\dots,n-1$, *be a Hankel matrix. Then*

$$L = W(y)AW^T(z) \tag{27}$$

is a Loewner matrix in $\mathscr{L}(y,z)$ *whose parameters* $c_1,\dots,c_n,d_1,\dots,d_n$ *are given (up to an arbitrary additive constant* ξ*) by*

$$\begin{pmatrix} c \\ d \end{pmatrix} = W(y,z)\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{2n-2} \\ \xi \end{pmatrix}, \qquad c=(c_1,\dots,c_n)^T, \quad d=(d_1,\dots,d_n)^T. \tag{28}$$

Conversely, if $L \in \mathscr{L}(y,z)$ *corresponds to* c_i, d_j, *then*

$$\begin{aligned} A &= [W(y)]^{-1}L[W^T(z)]^{-1} \\ &= V^T(y)D^{-1}(y)LD^{-1}(z)V(z) \end{aligned}$$

is a Hankel matrix: $A=(\alpha_{i+k})$, $i,k=0,\dots,n-1$, *where for some* ξ,

$$\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{2n-2} \\ \xi \end{pmatrix} = V^T(y,z)\Delta^{-1}\begin{pmatrix} c \\ d \end{pmatrix},$$

$$\Delta = \begin{pmatrix} D(y)H(y) & 0 \\ 0 & D(z)G(z) \end{pmatrix},$$

$$H(y) = \operatorname{diag}(h(y_i)), \qquad G(z) = \operatorname{diag}(g(z_i)),$$

$$g(x) = \prod(x-y_i), \qquad h(x) = \prod(x-z_i).$$

REMARK. It can be shown that a completely analogous theorem holds if A (and L) are not square matrices.

As is well known, the Cauchy interpolation problem in the simplest case means, given mutually distinct (complex) numbers $x_0, \dots, x_{2n}$ and (complex) numbers $f_0, \dots, f_{2n}$, to find polynomials $P(x), Q(x)$ of degree at most n such that $[Q(x_i) \neq 0$ and$]$

$$\frac{P(x_i)}{Q(x_i)} = f_i, \qquad i = 0, \dots, 2n.$$

To show the connection with the Loewner matrices, let us first say that a $2n$-tuple of points $(x_1, f_1), \dots, (x_{2n}, f_{2n})$ is nonsingular if the $2n \times 2n$ matrix

$$E(x, f) = \left(1, x_i, \dots, x_i^{n-1}, f_i, f_i x_i, \dots, f_i x_i^{n-1}\right) \tag{29}$$

(its ith row is written here) is nonsingular. The following theorem is essentially due to Loewner [11]; we shall present a short proof via the Schur complement [7]:

THEOREM 13. *Let $y_1, \dots, y_n, z_1, \dots, z_n$ be distinct; then the $2n$-tuple $(y_1, c_1), \dots, (y_n, c_n), (z_1, d_1), \dots, (z_n, d_n)$ is nonsingular iff the Loewner matrix $((c_i - d_j)/(y_i - z_j))$ is nonsingular.*

Proof. Since the corresponding matrix E can be written as

$$E = \begin{pmatrix} V(y) & CV(y) \\ V(z) & DV(z) \end{pmatrix}$$

with $C = \operatorname{diag}(c_i)$, $D = \operatorname{diag}(d_i)$, E is nonsingular iff the Schur complement $[E/V(y)]$ is nonsingular. However,

$$\begin{aligned}[E/V(y)] &= DV(z) - V(z)[V(y)]^{-1}CV(y) \\ &= -V(z)[V(y)]^{-1}H(y)L[W^T(z)]^{-1},\end{aligned}$$

where

$$L = \left(\frac{c_i - d_j}{y_i - z_j}\right)$$

and $H(y) = \operatorname{diag}(h(y_i))$, $h(x) = \prod(x - z_i)$. The result follows. ∎

THEOREM 14. *Let* $y_0, y_1, \dots, y_n, z_1, \dots, z_n$ *be mutually distinct complex numbers. Let* $c_0, \dots, c_n, d_1, \dots, d_n$ *be complex numbers; let* $P(x), Q(x)$ *be relatively prime polynomials with maximum degree* n. *Then the following are equivalent*:

(i) $Q(y_i) \neq 0$, $i = 0, \dots, n$, $Q(z_j) \neq 0$, $j = 1, \dots, n$, *and*

$$c_i = \frac{P(y_i)}{Q(y_i)}, \quad d_j = \frac{P(z_j)}{Q(z_j)} \qquad \text{for these } i, j.$$

(ii) *The polynomial*

$$R(x) = \det\left(\frac{c_i - d_j}{y_i - z_j}, g_i(x) \right)$$

with

$$g_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} (x - y_k)$$

is nonzero, and neither y_i *nor* z_j *is its root for* $i = 0, \dots, n$, $j = 1, \dots, n$; *moreover*,

$$Q(x) \equiv \rho R(x),$$

$$P(x) \equiv \rho \det\left(\frac{c_i - d_j}{y_i - z_j}, c_i g_i(x) \right)$$

for some $\rho \neq 0$.

(iii) *All* $2n$*-tuples obtained from*

$$(y_0, c_0), \dots, (y_n, c_n), (z_1, d_1), \dots, (z_n, d_n)$$

by deleting one of them are nonsingular; moreover, if we denote

$$E_1(x) = \det\begin{pmatrix} \tilde{E} \\ X_1 \end{pmatrix}, \qquad E_2(x) = \det\begin{pmatrix} \tilde{E} \\ X_2 \end{pmatrix}$$

where

$$\tilde{E}=\begin{pmatrix} 1 & y_i & \cdots & y_i^n & c_i & c_i y_i & \cdots & c_i y_i^n \\ 1 & z_j & \cdots & z_j^n & d_j & d_j z_j & \cdots & d_j z_j^n \end{pmatrix},$$

$$i=0,\ldots,n, \quad j=1,\ldots,n,$$

$$X_1=(1,x,\ldots,x^n,0,\ldots,0), \qquad X_2=(0,0,\ldots,0,1,x,\ldots,x^n),$$

then

$$E_2(y_i)\neq 0, \quad E_2(z_j)\neq 0, \qquad i=0,\ldots,n, \quad j=1,\ldots,n, \tag{30}$$

$$E_1(y_i)+c_iE_2(y_i)=0, \qquad E_1(z_j)+d_jE_2(z_j)=0, \tag{31}$$

$$P(x)=\sigma E_1(x), \quad Q(x)=-\sigma E_2(x) \qquad \textit{for some} \quad \sigma\neq 0. \tag{32}$$

Proof. (i)$\to$(ii): Since P,Q are relatively prime (i) implies that the matrix

$$\left(\frac{P(y_i)Q(z_j)-P(z_j)Q(y_i)}{y_i-z_j}\right), \qquad i=0,\ldots,n, \quad j=1,\ldots,n, \quad i\neq r,$$

is nonsingular, since it is of the form

$$V(y)B(P,Q)V^T(z)$$

by Theorem 3. Thus all the matrices obtained from the $(n+1)\times n$ matrix $((c_i-d_j)/(y_i-z_j))$ by deleting one row are nonsingular. It follows that $R(x)$ is a nonzero polynomial and does not have root y_i, $i=0,\ldots,n$.

Let us show now that for some $\mu\neq 0$, $R(x)=\mu\det(A,X)$, where $A=(\alpha_{i+k})$, $i=0,\ldots,n$, $k=1,\ldots,n$, $X=(1,x,\ldots,x^n)^T$, and

$$\begin{pmatrix} c \\ d \end{pmatrix}=W(y,z)\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{2n-1} \\ \xi \end{pmatrix}$$

with a similar notation to (28). Indeed, defining A by (28), we have by (27)

(for the rectangular case)

$$\left(\frac{c_i - d_j}{y_i - z_j}, g_i(x)\right) = \left(W(y)AW^T(z), W(y)X\right)$$

$$= W(y)(A, X)\begin{pmatrix} W^T(z) & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $R(x) = \mu \det(A, X)$ for $\mu = \det W(y) \det W^T(z) \neq 0$.

It follows from this observation that $R(x)$ changes only by a nonzero factor if we exchange, for $k \in \{1, \ldots, n\}$, z_k and y_0 and simultaneously d_k and c_0. Since y_0 was proved not to be a root of $R(x)$, the same is true of z_k, $k = 1, \ldots, n$.

Let us show that $R(x)$ and $S(x)$ defined by

$$S(x) = \det\left(\frac{c_i - d_j}{y_i - z_j}, c_i g_i(x)\right)$$

satisfy

$$\frac{S(y_i)}{R(y_i)} = c_i, \quad \frac{S(z_j)}{R(z_j)} = d_j, \qquad i = 0, \ldots, n, \quad j = 1, \ldots, n.$$

This is clear in the first case. For proving the second, observe that

$$R(z_k) = -g(z_k)\det\left(\frac{c_i - d_j}{y_i - z_j}, \frac{1}{y_i - z_k}\right)$$

while

$$S(z_k) = -g(z_k)\det\left(\frac{c_i - d_j}{y_i - z_j}, \frac{c_i}{y_i - z_k}\right),$$

so that

$$S(z_k) - d_k R(z_k) = -g(z_k)\det\left(\frac{c_i - d_j}{y_i - z_j}, \frac{c_i - d_k}{y_i - z_k}\right),$$

which is zero.

It follows that $R(x)P(x)-S(x)Q(x)$ is divisible by $\Pi_{i=0}^{n}(x-y_i)\Pi_{j=1}^{n}(x-z_j)$ and is thus zero. Since $P(x)$ and $Q(x)$ are relatively prime and their maximum degree is n, the assertion follows.

(ii) $\to$ (iii): As we have seen, $R(x)$ is a multiple of $\det(A, X)$, where A is defined independently of the choice of the $2n$-tuple of the $2n+1$ pairs $(y_i, c_i), (z_j, d_j)$. By (ii), the matrix

$$\left(\frac{c_i - d_j}{y_i - z_j}\right), \qquad i, j = 1, \dots, n,$$

is nonsingular. Theorem 13 implies that the $2n$-tuple obtained by deleting (y_0, c_0) is nonsingular. The above mentioned independence of $R(x)$ yields then the same for the remaining $2n$-tuples.

Now, an easy application of determinantal rules gives

$$E_2(y_0) = \det\begin{pmatrix} 1 & y_0 & \cdots & y_0^n & 0 & \cdots & 0 \\ 1 & y_i & \cdots & y_i^n & c_i & \cdots & c_i y_i^n \\ 1 & z_j & \cdots & z_j^n & d_j & \cdots & d_j z_j^n \\ 0 & 0 & \cdots & 0 & 1 & \cdots & y_0^n \end{pmatrix},$$

$$E_1(y_0) = c_0 \det\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & y_0^n \\ 1 & y_i & \cdots & y_i^n & c_i & \cdots & c_i y_i^n \\ 1 & z_j & \cdots & z_j^n & d_j & \cdots & d_j z_j^n \\ 1 & y_0 & \cdots & y_0^n & 0 & \cdots & 0 \end{pmatrix}.$$

Thus

$$E_1(y_0) + c_0 E_2(y_0) = 0. \tag{33}$$

To show that $E_2(y_0) \neq 0$, observe that subtracting from the kth column the $(-y_0)$-tuple of the $(k-1)$th column for $k = 2n+1, 2n, \dots, n+3, n+1, n, \dots, 2$ yields easily

$$E_2(y_0) = (-1)^n \prod_{i=1}^{n} (y_i - y_0) \prod_{j=1}^{n} (z_j - y_0) \det M,$$

where M is the matrix (29) for the remaining $2n$-tuple. Thus $E_2(y_0) \neq 0$, the same holds for $y_1, \dots, y_n, z_1, \dots, z_n$ and by (33), the same argument as in (i) $\to$ (ii) completes the proof of (ii) $\to$ (iii).

(iii) $\to$ (i): Follows immediately from (30), (31), and (32). ∎

Let us state a theorem which also shows a connection between the Loewner and Bézout matrices.

THEOREM 15. *Let $u(x)$, $v(x)$ be polynomials,* $\max(\deg u, \deg v) = n$. *Let*

$$L = \left(\frac{c_i - d_j}{y_i - z_j} \right) \in \mathscr{L}(y, z),$$

where $u(y_i) \neq 0$, $u(z_j) \neq 0$ and

$$c_i = \frac{v(y_i)}{u(y_i)}, \quad d_j = \frac{v(z_j)}{u(z_j)}, \qquad i, j = 1, \dots, n. \tag{34}$$

Then for

$$\Delta_1 = \operatorname{diag}(u(y_i)), \qquad \Delta_2 = \operatorname{diag}(u(z_j)) \tag{35}$$

we have

$$L = -\Delta_1^{-1} V(y) B(u, v) V^T(z) \Delta_2^{-1}. \tag{36}$$

REMARK. If $u(x)$ has degree n, one can assume that $v(x)$ has degree $\leqslant n-1$. We shall denote then by $L_{v/u}$ the matrix (34) from $\mathscr{L}(y, z)$.

The following interesting result is due to Vavřín [14]:

THEOREM 16. *Let $u(x)$, $v(x)$ be relatively prime polynomials,* $\deg u = n$, $\deg v \leqslant n-1$. *Then, $D(y)$, $D(z)$ being defined as in* (8) *and Δ_1, Δ_2 as in* (35),

$$L_{v/u}^{-1} = D^{-1}(z) \Delta_2 L_{w/u}^T \Delta_1 D^{-1}(y),$$

where $w(x)$ is that (unique) polynomial of degree less than n for which

$$v(x) w(x) \equiv g(x) h(x) \bmod u(x),$$

$$g(x) = \prod_i (x - y_i), \qquad h(x) = \prod_j (x - z_j).$$

To show that any nonsingular Loewner matrix has the form (34), we shall need a lemma.

LEMMA. *If* (x_i, f_i), $i = 1,\dots,2n$, *is a nonsingular* $2n$*-tuple, then there exists* (x_0, f_0) *such that any* $2n$*-tuple of the* $2n+1$ *points* (x_j, f_j), $j = 0,\dots,2n$, *is nonsingular.*

Proof. If this were not true, there would exist a $(2n-1)$-tuple say (x_i, f_i), $i = 1,\dots,2n-1$, such that both

$$\Phi_1(x) = \det\begin{pmatrix} 1 & x & \cdots & x^{n-1} & 0 & 0 & \cdots & 0 \\ 1 & x_i & \cdots & x_i^{n-1} & f_i & f_i x_i & \cdots & f_i x_i^{n-1} \end{pmatrix},$$

$$\Phi_2(x) = \det\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & x & \cdots & x^{n-1} \\ 1 & x_i & \cdots & x_i^{n-1} & f_i & f_i x_i & \cdots & f_i x_i^{n-1} \end{pmatrix}$$

are identically zero. However, the nonsingularity of the given n-tuple implies that

$$\Phi_1(x_{2n}) + f_{2n}\Phi_2(x_{2n}) \neq 0,$$

a contradiction. ■

THEOREM 17. *Let* (x_i, f_i), $i = 1,\dots,2n$, *be a* $2n$*-tuple, denoted by* S, *for which* x_i *are mutually distinct. Then the following are are equivalent*:

(i) S *is nonsingular*;

(ii) *for any decomposition of* S *into two n-tuples* $(y_i, c_i), (z_j, d_j)$, $i, j = 1,\dots,n$, *the Loewner matrix* $((c_i - d_j)/(y_i - z_j))$ *is nonsingular*;

(iii) *there exists a decomposition of* S *into two n-tuples as in* (ii) *for which the corresponding Loewner matrix is nonsingular*;

(iv) *there exist relatively prime polynomials* $u(x)$, $v(x)$ *with* $\max(\deg u, \deg v) = n$ *such that*

$$f_i = \frac{v(x_i)}{u(x_i)}, \qquad i = 1,\dots,2n.$$

Proof. The first three properties are equivalent by Theorem 13. To show that (i) → (iv), let (x_0, f_0) be a point from the lemma. Define $E_1(x)$, $E_2(x)$ as in (iii) of Theorem 14. As was shown in the proof of the implication (ii) → (iii),

$E_2(x_i) \neq 0$ for $i = 0,\ldots,2n$ and

$$E_1(x_i) + f_i E_2(x_i) = 0.$$

By Theorem 15 and (36), the Bézout matrix $B(u, v)$ for $u = E_2$, $v = -E_1$ is nonsingular. Thus E_1, E_2 are relatively prime and (iv) is proved.

(iv) $\to$ (iii) follows from Theorem 15. ■

Let us conclude with the remark that there is also a close connection between Loewner matrices and reciprocal differences [12]. This will be shown elsewhere.

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